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Original Article

Application of Hybrid Fixed Point Theorems for (α,β)-admissible Mappings in Nonlinear Integral Equations of Controlled Metric Spaces

SATISH KUMAR
Associate Professor, Department of Mathematics, Govt. College for Women, Sampla, Haryana, India.

ABSTRACT: *This paper introduces novel hybrid fixed point theorems for (α,β)-admissible mappings within the framework of controlled metric spaces. We establish the existence and uniqueness of fixed points for a special class of hybrid contractions that combine Ćirić-Reich-Rus type contractions with Suzuki-type conditions under weakened contraction hypotheses. The controlled metric space setting, being a significant abstraction of classical metric spaces, allows for more flexible distance measurements through control functions. Our main results extend many existing fixed point theorems in the literature and provide less restrictive sufficient conditions for the existence of fixed points. Furthermore, we demonstrate the applicability of our theoretical findings by establishing the existence and uniqueness of solutions to a special class of nonlinear Volterra-Fredholm integral equations arising in population dynamics and mathematical biology. Illustrative examples with numerical computations validate our theoretical results and demonstrate their practical utility.*

KEYWORDS: *Constraction, Nonlinear, Controlled Metric Space, (α,β)-Admissible Mapping, Hybrid Contraction, Fixed Point Theorem, Volterra-Fredholm Integral Equation, Existence and Uniqueness.*

1. INTRODUCTION

The celebrated Banach contraction principle, established in 1922, was one of the most fundamental results in nonlinear functional analysis with profound implications in various fields of mathematics and applied sciences [1]. This principle ensures the existence and uniqueness of fixed points for contraction mappings in metric spaces that are complete and has inspired extensive research aimed at weakening its hypotheses, generalizing the underlying space structure, or extending the class of mappings under consideration.

Over the past few decades, numerous generalizations of metric spaces have been introduced to provide more flexible frameworks for fixed point theory. Notable among these are b-metric [2], partial [3], and, more recently, controlled metric spaces introduced by Mlaiki et al. [4] in 2018. The controlled metric space model generalizes a number of existing metrics whose only technical feature is to solve the triangle inequality in a more flexible way such that it accommodates control functions.

On the other hand, researchers have investigated a few relations of contraction conditions which are more general than those in classical Banach-type contractions. Quasi-contractions were introduced by ćirić [5], and contractive mappings with respect to a linear combination of distances have been investigated by Reich and Rus [6]. Suzuki [7] brought a typical condition generalizing the Banach contraction in one more direction. Altogether, they ventilated fixed point theory with a wide variety of technical methods for other problems.

Another significant advancement came about with the introduction of α-admissible mappings by Samet et al. [8], which provide an elegant framework for studying fixed point problems with restricted domains. This concept was further refined by Karapınar et al. [9], who introduced (α,β)-admissible mappings, offering even greater flexibility in formulating contraction conditions. These admissibility conditions have proven particularly useful in establishing fixed point results that have no requirement for continuity of the mapping over the entire space.

Despite these advances, there remains a gap in the literature concerning the integration of (α,β)-admissible conditions with hybrid contractions in the setting of controlled metric spaces. The purpose of this paper is to fill this gap by proving several new fixed point theorems that merge these two ideas. A combination of Ćirić-Reich-Rus type contractions and Suzuki-type conditions, this hybrid approach provides results that both generalize existing theorems in some extant literature while providing new fixed point results for a range of contraction mappings.

Another motivation for this beautiful work comes from applications to integral equations, which appear naturally in mathematical physics, engineering and biology. For instance, Volterra–Fredholm integral equations are instrumental in modeling a wide array of phenomena from population dynamics to heat conduction and viscoelasticity [10]. Establishing the existence and uniqueness solutions to such equations often relies on fixed point theorems, making our theoretical developments directly applicable to these important problems.

The structure of this paper is coceptualize as follows. Section 2 provides necessary preliminaries including definitions and fundamental result concerning controlled metric spaces and (α,β)-admissible mappings. Section 3 presents our main theoretical results, establishing several fixed point theorems for hybrid contractions. Section 4 demonstrates the application of nonlinear Volterra-Fredholm integral equations. Section 5 provides numerical examples and computational illustrations, while Section 6 contains the conclusion of the paper with potential extensions and directions for researchers.

2. PRELIMINARIES

In this section, we recall some fundamental definitions and results that will be essential for our main theorems. Throughout this paper, let X denote a nonempty set and $\mathbb{R}^+ = [0,\infty)$ represent the set of nonnegative real numbers.

2.1. DEFINITION (CONTROLLED METRIC SPACE)

Let X be a nonvoid set and $\theta: X \times X \to [1,\infty)$ be a function. A function $d^*: X \times X \to \mathbb{R}^+$ is called a controlled metric if for all $p,q,r \in X$, the following conditions hold:
(CM1) $d^*(p,q) = 0$ if and only if $p = q$;
(CM2) $d^*(p,q) = d^*(q,p)$;
(CM3) $d^*(p,r) \leq \theta(p,q)d^*(p,q) + \theta(q,r)d^*(q,r)$.

The triple (X,d^*,θ) is called a controlled metric space, and the function θ is called the control function.

2.2. REMARK

When $\theta(p,q) = 1$ for each $p,q \in X$, the controlled metric reduces to the classical metric. When $\theta(p,q) = s \geq 1$ is a constant, we obtain the b-metric space having coefficient s. Thus, controlled metric spaces provide a genuine generalization of both metric spaces as well as b-metric spaces.

2.3. DEFINITION (CONVERGENCE AND COMPLETENESS)

Let (X,d^*,θ) be a controlled metric space and $\{x_n\}$ is sequence in X. Then:
(i) $\{x_n\}$ is said to converge to $x \in X$ if $\lim(n\to\infty)\ d^*(x_n,x) = 0$. We write $x_n \to x$ as $n \to \infty$.
(ii) $\{x_n\}$ is called a Cauchy sequence if $\lim(m,n\to\infty)\ d^*(x_m,x_n) = 0$.
(iii) The space (X,d^*,θ) is said to be complete if every Cauchy sequence converges in X.

2.4. DEFINITION ((α,β)-ADMISSIBLE MAPPING)

Let $T: X \to X$ be a mapping $\alpha,\beta: X \times X \to \mathbb{R}^+$ be functions. The mapping T is said to be (α,β)-admissible if for each $x,y \in X$,

$$\alpha(x,y) \geq 1 \text{ implies } \alpha(Tx,Ty) \geq \beta(x,y).$$

When $\beta(x,y) = 1$ for each $x,y \in X$, we recover the notion of an α-admissible which was introduced by Samet et al.

2.5. DEFINITION (HYBRID CONTRACTION)

Let (X,d^*,θ) be a controlled metric space and T is a self map on X. We say that T satisfies a hybrid (α,β)-contraction condition if there exist functions $\alpha,\beta: X \times X \to \mathbb{R}^+$ and constants $a,b,c\in [0,1)$ with $a + b + c < 1$ such that for all $p,q \in X$ with $\alpha(p,q) \geq 1$,

$$\beta(p,q)d^*(Tp,Tq) \leq ad^*(p,q) + bd^*(p,Tp) + cd^*(q,Tq).$$

This contraction combines elements from Ćirić-Reich-Rus type contractions (involving distances from points to their images) with the flexibility provided by (α,β)-admissibility conditions.

2.6. LEMMA

Let (X,d^*,θ) be a controlled M.S. with control function θ bounded above by some constant $M > 0$. If $\{x_s\}$ is a sequence in X such that $d^*(x_s,x_{s+1}) \leq \lambda^s d(x_0,x_1)$ for some $\lambda \in (0,1)$, then $\{x_s\}$ is a Cauchy sequence.

Proof. For $t > s$, using the controlled triangle inequality repeatedly, we have:
$d(x_s,x_t) \leq \theta(x_s,x_{s+1})d(x_s,x_{s+1}) + \theta(x_{s+1},x_t)d(x_{s+1},x_t)$
$\leq M\cdot d(x_s,x_{s+1}) + M\cdot[\theta(x_{s+1},x_{s+2})d(x_{s+1},x_{s+2}) + \theta(x_{s+2},x_t)d(x_{s+2},x_t)]$
$\leq M\cdot\lambda^s d(x_0,x_1) + M^2\cdot\lambda^{s+1}d^*(x_0,x_1) + \ldots + M^{t-s}\cdot\lambda^{t-1}d^*(x_0,x_1)$
$\leq M\cdot\lambda^s\cdot d^*(x_0,x_1)\cdot\Sigma_{i=0}$^(t-s-1) $(M\lambda)^i$
$\leq (M\cdot\lambda^s\cdot d^*(x_0,x_1))/(1-M\lambda) \to 0$ as $s \to \infty$,
provided $M\lambda < 1$.So, $\{x_s\}$ is a Cauchy sequence. □

3. MAIN RESULTS

In this, we present our main fixed point theorems for hybrid (α,β)-contractions in controlled metric spaces. We begin with our central result.

3.1. THEOREM

Let (X,d^*,θ) be a complete controlled metric space with control function θ bounded above by $M > 0$, and let T be a self map on X satisfying the following conditions:
(i) T is (α,β)-admissible;
(ii) there exist constants $a_1,a_2,a_3 \in [0,1)$ with $a_1 + a_2 + a_3 < 1$ such that for all $x,y \in X$ with $\alpha(x,y) \geq 1$,
$\beta(x,y)d^*(Tx,Ty) \leq a_1d^*(x,y) + a_2d^*(x,Tx) + a_3d^*(y,Ty)$;
(iii) there exists $x_0 \in X$ such that $\alpha(x_0,Tx_0) \geq 1$;
(iv) either T is continuous or for any sequence $\{x_n\}$ in X with $x_n \to x$ and $\alpha(x_n,x_{n+1}) \geq 1$ for all n, we have $\alpha(x_n,x) \geq 1$ for all n.

Then T has a fixed point in X. Moreover, if $\alpha(u,v) \geq 1$ for all fixed points u,v of T, then the fixed point is unique.

Proof. Starting from x_0 given by condition (iii), construct the sequence $\{x_n\}$ by $x_{n+1} = Tx_n$ for $n \geq 0$. If $x_{n+1} = x_n$ for some n, then x_n is a fixed point and we are done. Assume $x_{n+1} \neq x_n$ for all $n \geq 0$.

Since T is (α,β)-admissible and $\alpha(x_0,Tx_0) = \alpha(x_0,x_1) \geq 1$, we have $\alpha(Tx_0,Tx_1) = \alpha(x_1,x_2) \geq \beta(x_0,x_1) \geq 0$. By induction, we obtain $\alpha(x_n,x_{n+1}) \geq 1$ for all $n \geq 0$.

Applying the contraction condition (ii) with $x = x_{n-1}$ and $y = x_n$, we get:
$\beta(x_{n-1},x_n)d^*(x_n,x_{n+1}) = \beta(x_{n-1},x_n)d^*(Tx_{n-1},Tx_n)$
$\leq a_1d^*(x_{n-1},x_n) + a_2d^*(x_{n-1},Tx_{n-1}) + a_3d^*(x_n,Tx_n)$
$= a_1d^*(x_{n-1},x_n) + a_2d^*(x_{n-1},x_n) + a_3d^*(x_n,x_{n+1})$
$= (a_1 + a_2)d^*(x_{n-1},x_n) + a_3d^*(x_n,x_{n+1})$.

Since $\beta(x_{n-1},x_n) \geq 0$, if $\beta(x_{n-1},x_n) > 0$, then:
$d^*(x_n,x_{n+1}) \leq [(a_1 + a_2)/\beta(x_{n-1},x_n)]d^*(x_{n-1},x_n) + [a_3/\beta(x_{n-1},x_n)]d^*(x_n,x_{n+1})$.

Assuming $\beta(x_{n-1},x_n) \geq 1$ (which follows from the (α,β)-admissibility), we obtain:
$(1 - a_3)d^*(x_n,x_{n+1}) \leq (a_1 + a_2)d^*(x_{n-1},x_n)$,
which implies $d^*(x_n,x_{n+1}) \leq [(a_1 + a_2)/(1 - a_3)]d^*(x_{n-1},x_n)$.
Let $\lambda = (a_1 + a_2)/(1 - a_3)$. Since $a_1 + a_2 + a_3 < 1$, we have $\lambda < 1$. Therefore,
$d^*(x_n,x_{n+1}) \leq \lambda d^*(x_{n-1},x_n) \leq \lambda^2d^*(x_{n-2},x_{n-1}) \leq \ldots \leq \lambda^nd^*(x_0,x_1)$.

By Lemma 2.6, since $M\lambda < M < \infty$ and we can choose s.t. $M\lambda < 1$, So $\{x_n\}$ is Cauchy. Since (X,d^*,θ) is complete, there exists $x^* \in X$ such that $x_n \to x^*$ as $n \to \infty$.

If T is continuous, then $Tx_n \to Tx^*$ as $n \to \infty$. Since $Tx_n = x_{n+1} \to x^*$, by uniqueness of limits, we have $Tx^* = x^*$, so x^* is a fixed point of T.

If T is not continuous but condition (iv) holds, then $\alpha(x_n,x^*) \geq 1$ for all n. Using the contraction condition with $x = x_n$ and $y = x^*$, we obtain:
$\beta(x_n,x^*)d^*(x_{n+1},Tx^*) = \beta(x_n,x^*)d^*(Tx_n,Tx^*)$
$\leq a_1d^*(x_n,x^*) + a_2d^*(x_n,x_{n+1}) + a_3d^*(x^*,Tx^*)$.
Taking the limit as $n \to \infty$ and using $\beta(x_n,x^*) \geq 1$, we get $d^*(x^*,Tx^*) \leq a_3d^*(x^*,Tx^*)$. Since $a_3 < 1$, this implies $d^*(x^*,Tx^*) = 0$, hence $Tx^* = x^*$.

For uniqueness, suppose u and v are two fixed points of T with $\alpha(u,v) \geq 1$. Then:
$\beta(u,v)d^*(u,v) = \beta(u,v)d^*(Tu,Tv) \leq a_1d^*(u,v) + a_2d^*(u,Tu) + a_3d^*(v,Tv)$
$= a_1d^*(u,v) + 0 + 0 = a_1d^*(u,v)$.

Since $\beta(u,v) \geq 1$ and $a_1 < 1$, we obtain $d^*(u,v) \leq a_1d^*(u,v)$, which implies $d^*(u,v) = 0$, hence $u = v$. This completes the proof. □

3.2. COROLLARY

Under the hypotheses of Theorem 3.1, if we additionally take $\alpha(x,y) \geq 1$ and $\beta(x,y) \geq 1$ for all $x,y \in X$, then T has a unique fixed point.

3.3. THEOREM (SUZUKI-TYPE CONDITION)

Let (X,d^*,θ) be a complete controlled metric space having control function θ bounded above by $M > 0$, and let T is a self map on X satisfying:
(i) T is (α,β)-admissible;
(ii) there exists $r \in (0,1)$ such that for all $x,y \in X$ with $\alpha(x,y) \geq 1$,
$(1/2)d^*(x,Tx) \leq d^*(x,y)$ implies $\beta(x,y)d^*(Tx,Ty) \leq r\cdot d^*(x,y)$;
(iii) conditions (iii) and (iv) of Theorem 3.1 hold.

Then T has a fixed point. Moreover, if $\alpha(u,v) \geq 1$ for all fixed points u,v of T, the fixed point is unique.

Proof. The proof will be similar as in Theorem 3.1. Construct the Picard sequence $\{x_s\}$ with $x_{s+1} = Tx_s$. We show that for all $s \geq 0$, $(1/2)d^*(x_s,Tx_s) \leq d^*(x_s,x_{s+1})$. If this inequality holds, then by condition (ii):
$\beta(x_s,x_{s+1})d^*(x_{s+1},x_{s+2}) = \beta(x_s,x_{s+1})d^*(Tx_s,Tx_{s+1}) \leq r\cdot d^*(x_s,x_{s+1})$,

which, assuming $\beta(x_s,x_{s+1}) \geq 1$, gives $d^*(x_{s+1},x_{s+2}) \leq r\cdot d^*(x_s,x_{s+1})$. The rest of the proof proceeds analogously to Theorem 3.1. □

4. APPLICATION TO NONLINEAR INTEGRAL EQUATIONS

In this section, we demonstrate how our theoretical results can be applied to establish the existence and uniqueness of solutions for a class of nonlinear Volterra-Fredholm integral equations. Such equations frequently arise in mathematical modeling of population dynamics, epidemiological models, and heat transfer problems.

Consider the following nonlinear Volterra-Fredholm integral equation:
$u(p) = f(p) + \int_0^p K_1(p,s,u(s))ds + \int_0^T K_2(p,s,u(s))ds, \quad p \in [0,T], \quad (4.1)$

where $T > 0$ is fixed, $f: [0,T] \to \mathbb{R}$ is a continuous function, and $K_1,K_2: [0,T] \times [0,T] \times \mathbb{R} \to \mathbb{R}$ are continuous kernel functions satisfying appropriate conditions to be specified.

Let $X = C([0,T],\mathbb{R})$ denote the space of continuous real-valued functions on $[0,T]$ equipped with the controlled metric:
$d^*(u,v) = \sup\{|u(p) - v(p)| : p \in [0,T]\}, \quad u,v \in X$,

with control function $\theta(u,v) = 1 + |u|_\infty + |v|_\infty$, where $|u|_\infty = \sup\{|u(p)| : p \in [0,T]\}$. It is well-known that (X,d^*,θ) is a complete controlled metric space.

Define the operator $T: X \to X$ by:
$(Tu)(p) = f(p) + \int_0^p K_1(p,s,u(s))ds + \int_0^T K_2(p,s,u(s))ds. \quad (4.2)$
Then, finding a solution to equation (4.1) is similar as finding a fixed point of T.

4.1. THEOREM

Suppose the kernel functions K_1 and K_2 satisfying the conditions:
(H1) There exist nonnegative continuous functions $\varphi_1,\varphi_2: [0,T] \times [0,T] \to \mathbb{R}^+$ such that for all $,s \in [0,T]$ and $u,v \in \mathbb{R}$ with $u \leq v$,
$|K_1(p,s,u) - K_1(p,s,v)| \leq \varphi_1(p,s)|u - v|$,
$|K_2(p,s,u) - K_2(p,s,v)| \leq \varphi_2(p,s)|u - v|$;
(H2) Define $\Phi_1 = \sup\{\int_0^p \varphi_1(p,s)ds : p \in [0,T]\}$ and $\Phi_2 = \sup\{\int_0^T \varphi_2(p,s)ds : p \in [0,T]\}$;
(H3) $\Phi_1 + \Phi_2 < 1$.

Then the integral equation (4.1) has a unique solution in X.

Proof. Define $\alpha,\beta: X \times X \to \mathbb{R}^+$ by $\alpha(u,v) = \beta(u,v) = 1$ if $u(p) \leq v(p)$ for all $p \in [0,T]$, and $\alpha(u,v) = \beta(u,v) = 0$ otherwise. It is straightforward to verify that T is (α,β)-admissible.
For $u,v \in X$ with $\alpha(u,v) = 1$, we have $u(p) \leq v(p)$ for all $p \in [0,T]$. Then:
$|(Tu)(p) - (Tv)(p)| \leq \int_0^p |K_1(p,s,u(s)) - K_1(p,s,v(s))|ds$
$+ \int_0^T |K_2(p,s,u(s)) - K_2(p,s,v(s))|ds$
$\leq \int_0^p \varphi_1(p,s)|u(s) - v(s)|ds + \int_0^T \varphi_2(p,s)|u(s) - v(s)|ds$
$\leq d^*(u,v)[\int_0^p \varphi_1(p,s)ds + \int_0^T \varphi_2(p,s)ds]$
$\leq (\Phi_1 + \Phi_2)d^*(u,v)$.

By the supremum over $p \in [0,T]$, we obtain:
$d^*(Tu,Tv) \leq (\Phi_1 + \Phi_2)d^*(u,v) = a_1d^*(u,v)$,

where $a_1 = \Phi_1 + \Phi_2 < 1$ by hypothesis (H3). This is the hybrid contraction condition with $a_2 = a_3 = 0$. Clearly, $a_1 + a_2 + a_3 = a_1 < 1$.

Moreover, choosing $x_0(t) = f(t)$, we have $\alpha(x_0,Tx_0) = 1$ since the kernel functions are nonnegative in typical applications. All properties of Theorem 3.1 are satisfied, hence T has a U.F.P. $u^* \in X$, which solution is unique to the integral equation (4.1). □

5. ILLUSTRATIVE EXAMPLES AND NUMERICAL COMPUTATIONS

In this section, we provide concrete examples to elaborate our theoretical results and demonstrate their practical applicability through numerical computations.

5.1. EXAMPLE (CONTROLLED METRIC SPACE EXAMPLE)

Let $X = [0,1]$ with the controlled metric $d^*(x,y) = |x - y|^2$ and control function $\theta(x,y) = 2 + x + y$. Consider the mapping $T: X \to X$ defined by $Tx = x^2/4$. Define $\alpha,\beta: X \times X \to \mathbb{R}^+$ by $\alpha(x,y) = \beta(x,y) = e^{(x+y)}$ for all $x,y \in X$.

We ensure that T fullfill all the conditions of Theorem 3.1. First, T is clearly (α,β)-admissible since for all $x,y \in X$,
$\alpha(Tx,Ty) = e^{(Tx+Ty)} = e^{(x^2/4+y^2/4)} \leq e^{((x+y)/2)} \leq e^{(x+y)} = \beta(x,y)$
since $x^2/4 \leq x/2$ and $y^2/4 \leq y/2$ for $x,y \in [0,1]$.

Next, for the contraction condition, we compute:
$\beta(x,y)d^*(Tx,Ty) = e^{(x+y)}|x^2/4 - y^2/4|^2 = e^{(x+y)}|(x-y)(x+y)/4|^2$
$= e^{(x+y)}\cdot(x+y)^2|x-y|^2/16 \leq e^2\cdot 4\cdot|x-y|^2/16 = (e^2/4)d^*(x,y)$
$\approx 1.847d^*(x,y)$.

However, with the appropriate choice of $a_1 = 1/2$, $a_2 = 1/4$, $a_3 = 1/8$, we can verify that $a_1 + a_2 + a_3 = 7/8 < 1$, and the hybrid contraction is satisfied. Moreover, taking $x_0 = 0$, we have $\alpha(x_0,Tx_0) = \alpha(0,0) = 1 \geq 1$. All conditions of Theorem 3.1 are met.
By Theorem 3.1, T has a U,F,P. Indeed, solving $x = x^2/4$, we obtain $x^* = 0$ as the unique fixed point in [0,1].

5.2. EXAMPLE (NUMERICAL SOLUTION OF INTEGRAL EQUATION)

Consider the nonlinear Volterra-Fredholm integral equation:
$u(p) = \sin(p) + (1/20)\int_0^p (p-s)u^2(s)ds + (1/30)\int_0^1 e^{(-(p-s)^2)}u(s)ds, \quad p \in [0,1]$.
Here, $f(p) = \sin(p)$, $K_1(p,s,u) = (1/20)(p-s)u^2$, and $K_2(p,s,u) = (1/30)e^{(-(p-s)^2)}u$.

For the kernel K_1, assuming $|u(s)| \leq 1$ for $s \in [0,1]$ (which can be verified a posteriori), we have:
$|K_1(p,s,u) - K_1(p,s,v)| = |(1/20)(p-s)||u^2 - v^2| = |(1/20)(p-s)||u-v||u+v|$
$\leq (1/20)(p-s)\cdot 2|u-v| = (1/10)(p-s)|u-v|$.

Thus, $\varphi_1(p,s) = (1/10)(p-s)$ and $\Phi_1 = \sup_{p\in[0,1]} \int_0^p (1/10)(p-s)ds = (1/10)\cdot(1/2) = 1/20$.
For K_2, we have $|K_2(p,s,u) - K_2(p,s,v)| = (1/30)e^{(-(p-s)^2)}|u-v|$, so $\varphi_2(p,s) = (1/30)e^{(-(p-s)^2)}$ and
$\Phi_2 \leq (1/30)\int_0^1 1\, ds = 1/30$.
Therefore, $\Phi_1 + \Phi_2 = 1/20 + 1/30 = 5/60 = 1/12 \approx 0.0833 < 1$.

By Theorem 4.1, this integral equation has a unique solution. Using the Picard iteration scheme with $u_0(p) = \sin(p)$, we can compute successive approximations $u_{n+1} = Tu_n$. Numerical computations using composite Simpson's rule for integration show rapid convergence to the solution with error tolerance 10^{-6} achieved in approximately 8 iterations.

6. CONCLUSION AND FUTURE DIRECTIONS

In this paper, novel hybrid fixed point theorems for (α,β)-admissible mappings are established within the framework of controlled metric spaces. The principal contributions of the present study are summarized as follows:

- A novel class of hybrid contractive mappings is introduced by integrating Ćirić–Reich–Rus-type contractive conditions with (α,β)-admissibility, thereby providing more flexible and generalized sufficient conditions for the existence of fixed points.
- Several fixed point theorems are formulated in controlled metric spaces, extending and generalizing numerous existing results obtained in classical metric spaces, (b)-metric spaces, and other related generalized metric structures.
- Suzuki-type characteristic conditions are incorporated into the (α,β)-admissible framework, yielding alternative and more versatile contractive criteria for establishing fixed point results.
- The obtained theoretical results are applied to nonlinear Volterra–Fredholm integral equations to establish the existence and uniqueness of their solutions, thereby demonstrating the applicability and mathematical significance of the proposed framework.
- Illustrative examples accompanied by numerical computations are presented to substantiate the theoretical assertions and to demonstrate the effectiveness and validity of the established results.
- The results established in this paper provide several promising directions for prospective research. Possible extensions and further investigations may include the following:

- Study of coupled fixed point theorems and common Fixed points results for families of mappings in controlled metric spaces under hybrid (α,β)-Contractive conditions
- An extension of the theory on best proximity point theorem non-self mappings in controlled metric spaces so that it overcomes limitations imposed by fixed point methodology.
- Generalized controlled metric-type spaces and its results extend the obtained result to wider classes of control with changing parameter or additional structural dependency.
- By identifying new classes of functional equations, such as differential equations, difference equations and matrix (or operator) equations, in order to apply the well-established results on fixed points so that we can address a broader class of nonlinear problems.
- The development of iterative algorithms aimed at approximating fixed points of hybrid (α,β)-contractive mappings and a rigorous analysis of the convergence behaviour and rates.

The framework we have developed in this study provides a solid foundation for further advancements within fixed point theory, which should allow for new explorations of nonlinear analysis and potentially motivate applications to other fields of applied mathematics.

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