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Original Article

Total Edge Irregularity Strength of Some Graphs

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ABSTRACT: *An edge irregular total k-labeling $f : V \cup E \rightarrow \{1,2,3,...,k\}$ of a graph G = (V,E) is a labeling of vertices and edges of G in such a way that for any two different edges uv and u^1v^1 their weight $f(u)+f(uv)+f(v)$ and $f(u_1)+f(u_1v_1)+f(v_1)$ are distinct. The total edge irregularity strength, tes(G), is defined as the minimum k for which the graph G has an edge irregular total k-labeling. In this paper, we study the total edge irregularity strength of joint-Wheel graph WH_n, < W_n : W_m > and path union of graph.*

KEYWORDS: *Irregularity Strength, Total Edge Irregularity Strength, Edge Irregular Total Labeling.*
***AMS Classification (2010)**: 05C78*

1. INTRODUCTION

All the graphs considered in this paper are simple, finite and undirected. In [3] Bača et.al. introduced the notion of an edge irregular total k-labeling of a graph *G(V,E)* to be a labeling of vertices and edges of G $\phi : V \cup E \rightarrow \{1,2,3,...,k\}$ such that the edge weights $wt_{\phi}(uv) = \phi(u)+\phi(uv)+\phi(v)$ are distinct for all edges that is, $wt_{\phi}(uv) \neq wt_{\phi}(u_1v_1)$ for every pair of edges $uv, u_1v_1 \in E$. The minimum k for which the graph G has an edge irregular total k-labeling is called the total edge irregularity strength of G, tes(G).They found a lower bound for the total edge irregularity strength of a graph as

$$\text{tes}(G) \geq max\left\{\left\lceil\frac{|E(G)|+2}{3}\right\rceil, \left\lceil\frac{\Delta(G)+1}{2}\right\rceil\right\} \ldots\ldots\ldots\ldots(1)$$

where $\Delta(G)$ is the maximum degree of G. Ivančo and Jendrol' posed the following conjecture.

Conjecture 1.1 [6] Let G be an arbitrary graph different from k_5 .Then

$$\text{tes}(G) = max\left\{\left\lceil\frac{|E(G)|+2}{3}\right\rceil, \left\lceil\frac{\Delta(G)+1}{2}\right\rceil\right\} \ldots\ldots\ldots(2)$$

Conjecture 1.1 has been verified by several authors for several families of graphs. Motivated by the results in [1,2,5,6,7,8,10,11,12,13] , we determine the exact values of the total edge irregularity strength of some graphs namely, joint-Wheel graph WH_n, < W_n : W_m > and path union of graph

Definition: 1.2 Joint-Wheel graph WH_n is consist of two disjoint copies of Wheel which are joined by an edge between two rim vertices. WH_n has 2n + 2 vertices and 4n + 1 edges, where n is the number of rim vertices in one copy of the Wheel graph.

Definition: 1.3 The graph G = < W_n : W_m > is the graph obtained by joining apex vertices of wheels W_n and W_m to a new vertex C.

Definition: 1.4 Let G be a graph and $G_1,G_2, . . . ,G_n$, $n \geq 2$ be n copies of graph G. The graph obtained by adding an edge from Gi to Gi+1(for i =1, 2, . . . , n − 1) is called path union of G.

2. MAINRESULTS

Theorem: 2.1 $tes(WH_n) = \left\lceil \frac{4n+3}{3} \right\rceil, n \geq 3.$

Proof:

$V(WH_n) = \{v_i, u_i, C_1, C_2\ 1 \leq i \leq n\}$ and $E(WH_n) = \{v_i v_{i+1}, u_i u_{i+1}, C_1 v_i, C_2 u_i, v_n u_n : 1 \leq i \leq n\}$ with indices taken modulo *n*.

Let $k = \left\lceil \frac{4n+3}{3} \right\rceil$, then from (1) it follows that $tes(WH_n) \geq \left\lceil \frac{|E(G)|+2}{3} \right\rceil = \left\lceil \frac{4n+3}{3} \right\rceil = k$, that is $tes(WH_n) \geq k$. Now to prove the reverse inequality ,we define the function f as follows.

$$f(c_1) = n, f(c_2) = \left\lceil \frac{k}{2} \right\rceil$$

$$f(v_i) = \left\lceil \frac{i}{2} \right\rceil, 1 \leq i \leq n; f(v_i v_{i+1}) = 2 + i - \left\lceil \frac{i}{2} \right\rceil - \left\lceil \frac{i+1}{2} \right\rceil, 1 \leq i \leq n;$$

$$f(c_1 v_i) = 2 + i - \left\lceil \frac{i}{2} \right\rceil, 1 \leq i \leq n; f(c_2 u_i) = 2n + 3 - k - \left\lceil \frac{k}{2} \right\rceil + i, 1 \leq i \leq n;$$

$$f(u_i u_{i+1}) = 3n + 3 - 2k + i, 1 \leq i \leq n; f(u_n v_n) = 2n + 3 - k - \left\lceil \frac{n}{2} \right\rceil.$$

We observe that, for $1 \leq i \leq n$;

$$wt(v_i v_{i+1}) = 2 + i, wt(c_1 v_i) = n + 2 + i, wt(v_n u_n) = 2n + 3,$$
$$wt(c_2 u_i) = 2n + 3 + i, wt(u_i u_{i+1}) = 3n + 3 + i.$$

It can be easily verified that all the vertex and edge labels are at most k and the weights of the edges are pair wise distinct. Thus the resulting total labeling is the edge irregular total k-labeling.

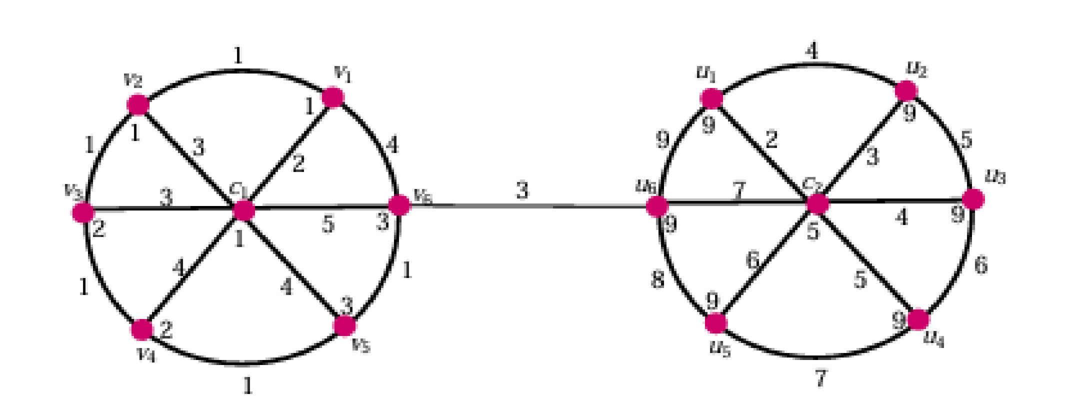

FIGURE 1 **tes (WH_6)=9**

Theorem: 2.2 $tes(\langle W_n : W_n \rangle) = \left\lceil \frac{4n+1}{3} \right\rceil, n \geq 3.$

Proof:

$V(\langle W_n : W_n \rangle) = \{v_i, u_i, C_1, C_2\ 1 \leq i \leq n\}$ and $E(\langle W_n : W_n \rangle) = \{v_i v_{i+1}, u_i u_{i+1}, c_1 v_i, c_2 u_i, c_1 c, c_2 c : 1 \leq i \leq n$ } with indices taken modulo *n*.

Let $k=\left\lceil \frac{4n+1}{3} \right\rceil$, then from (1) it follows that $tes(\langle W_n : W_n \rangle) \geq \left\lceil \frac{|E(G)|+2}{3} \right\rceil = \left\lceil \frac{4n+1}{3} \right\rceil = k$,that is $tes(\langle W_n : W_n \rangle) \geq k$ .Now to prove the reverse inequality ,we define the function f as follows.

$f(c_1)=n, f(c_2)=k, f(c)=\left\lceil \frac{k}{2} \right\rceil;$

$f(v_i)=\left\lceil \frac{i}{2} \right\rceil, 1\leq i \leq n; f(v_i v_{i+1}) = 2+i-\left\lceil \frac{i}{2} \right\rceil - \left\lceil \frac{i+1}{2} \right\rceil, 1 \leq i \leq n;$

$f(c_1 v_i) = 2+i-\left\lceil \frac{i}{2} \right\rceil, 1\leq i \leq n; f(c_1 c) = n+3-\left\lceil \frac{k}{2} \right\rceil; f(c_2 c) = 2n+4-k-\left\lceil \frac{k}{2} \right\rceil;$

$f(c_2 u_i) = \min\{2n+4-k+i, k\}$

- $for \quad 1 \leq i \leq n, such that\, 2n+4-k+i \leq k$

$f(u_i)=n$

- $for \quad 1 \leq i \leq n, such that\, 2n+4-k+i > k$

$f(u_i) = 3n+4-2k+i$

- $for \quad 1 \leq i \leq n-1, such that\, 2n+4-k+i < k$

$f(u_i u_{i+1}) = 4+i$

- $for \quad 1 \leq i \leq n-1, such that\, 2n+4-k+i = k$

$f(u_i u_{i+1}) = 2k-2n-1$

- $for \quad 1 \leq i \leq n-1, such that\, 2n+4-k+i > k$

$f(u_i u_{i+1}) = 4k-4n-4-i-1$

$f(u_n u_1) = 2k-2n.$

We observe that, for $1 \leq i \leq n$;

$wt(v_i v_{i+1}) = 2+i, wt(c_1 v_i) = n+2+i, wt(c_1 c) = 2n+3,$

$wt(cc_2) = 2n+4, wt(u_i u_{i+1}) = 2n+4+i, wt(c_2 u_i) = 3n+4+i.$

It can be easily verified that all the vertex and edge labels are at most k and the weights of the edges are pair wise distinct. Thus the resulting total labeling is the edge irregular total k-labeling.

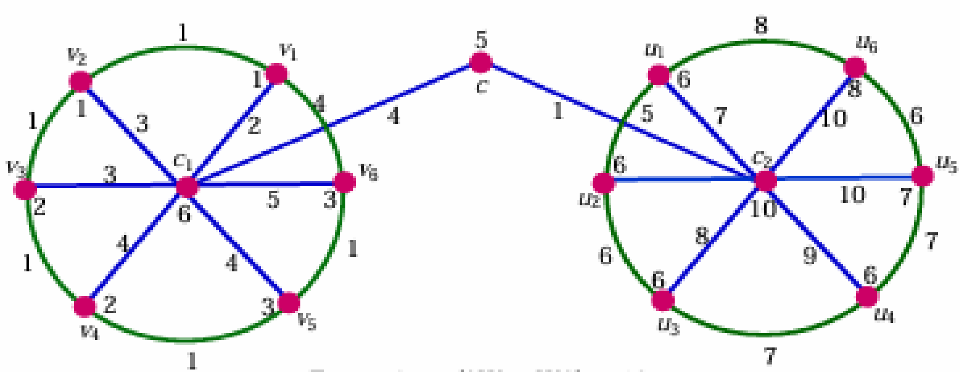

FIGURE 2 $tes(\langle W_6 : W_6 \rangle) = 10$

Theorem: 2.3 The path union of 2 copies of helm graph of total edge irregularity strength is 2n+1

Proof: Let G be the path union of 2 copies G_1, G_2 of helm graph H_n.

$$V(G)=\{v_i^1, u_i^1, v_i^2, u_i^2, c_1, c_2 : 1 \le i \le n\}$$

$$E(G)=\{c_1v_i^1, v_i^1v_{i+1}^1, v_i^1u_i^1, c_2v_i^2, v_i^2v_{i+1}^2, v_i^2u_i^2 : 1 \le i \le n\}.$$

$$tes(G) \ge \left\lceil \frac{|E(G)|+2}{3} \right\rceil = \left\lceil \frac{6n+3}{3} \right\rceil = 2n+1.$$

$$f(u_i^1)=1, f(v_i^1)=i, 1 \le i \le n; f(c_1)=n, f(c_2)=2n+1, f(u_i^2)=n, 1 \le i \le n;$$

$$f(v_i^2)=2n+1, 1 \le i \le n; f(u_i^1v_i^1)=1, 1 \le i \le n; f(c_1v_i^1)=2, 1 \le i \le n; f(c_1c_2)=2;$$

$$f(u_i^2v_i^2)=2+i, 1 \le i \le n; f(v_i^1v_{i+1}^1)=\begin{cases} 2n+1-i, & 1 \le i \le n-1 \\ 2n+1, & i=n. \end{cases}.$$

$$f(v_i^2v_{i+1}^2)=1+i, 1 \le i \le n; f(c_2v_i^2)=n+1+i, 1 \le i \le n;$$

We observe that, for $1 \le i \le n$

$$wt(u_i^1v_i^1)=2+i, 1 \le i \le n;$$

$$wt(c_1v_i^1)=n+2+i, 1 \le i \le n; wt(c_1c_2)=3n+3;$$

$$wt(u_i^2v_i^2)=3n+3+i, 1 \le i \le n;$$

$$wt(v_i^1v_{i+1}^1)=\begin{cases} 2n++2+i, & 1 \le i \le n-1 \\ 3n+2, & i=n. \end{cases}.$$

$$wt(v_i^2v_{i+1}^2)=4n+3+i, 1 \le i \le n; f(c_2v_i^2)=5n+3+i, 1 \le i \le n;$$

It can be easily verified that all the vertex and edge labels are at most 2n+1 and the weights of the edges are pair wise distinct. Thus the resulting total labeling is the edge irregular total (2n+1)-labeling

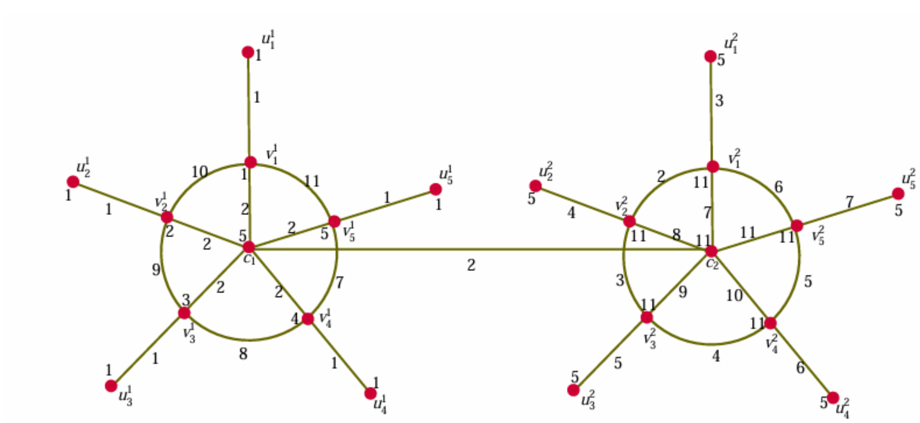

FIGURE 3 Total Edge Irregularity Strength of Path Union of Helm H_5 Graph is 11

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